Online Appendix to Albuquerque (2012)

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This note proves that unconditional skewness is positive in the model of Section 2 of Albuquerque (2012). I thank Johan Walden for providing me with the main details.

Proposition 1 *Unconditional skewness in firm stock returns is positive for $K \geq 1$.*

Proof. Consider the expression for unconditional skewness given in Corollary 2 of Albuquerque (2012),

$$E\left[(Q - E(Q_{t+1}))^3\right] = \frac{1}{K+1}\sum_{k=0}^{K}(\mu_k - E(Q_{t+1}))^3 + \frac{3}{(K+1)^2}\sum_{k=0}^{K}\sum_{j<k}\left(\sigma_k^2 - \sigma_j^2\right)\left(\mu_k - \mu_j\right), \quad (1)$$

and note that for all k, $\mu_k = \gamma\sigma_k^2$.

The second term on the right hand side of equation (1) is positive since

$$\sum_{k=0}^{K}\sum_{j<k}\left(\sigma_k^2 - \sigma_j^2\right)\left(\mu_k - \mu_j\right) = \gamma\sum_{k=0}^{K}\sum_{j<k}\left(\sigma_k^2 - \sigma_j^2\right)^2.$$

Turn now to the first term on the right hand side of equation (1) and write it as $\frac{1}{K+1}\sum_{k=0}^{K} a_k^3$, where $a_k = \mu_k - \frac{1}{K+1}\sum_{j=0}^{K}\mu_j$. Suppose $K = 1$. Then

$$\frac{1}{K+1}\sum_{k=0}^{K} a_k^3 = \frac{1}{2}\left(\mu_0 - \frac{1}{2}(\mu_0 + \mu_1)\right)^3 + \frac{1}{2}\left(\mu_1 - \frac{1}{2}(\mu_0 + \mu_1)\right)^3 = 0,$$

and skewness is positive. Suppose $K > 1$. I want to show that $\sum_{k=0}^{K} a_k^3 \geq 0$. The proof uses results from majorization theory. Define k_0 such that $a_k \leq 0$, for $k \leq k_0$, and $a_k > 0$, for $k > k_0$. Then, using the convexity of μ_k, and thus of a_k, and the fact that $\sum_{k=0}^{K} a_k = 0$, it must be that $k_0 + 1 \geq (K+1)/2$. Also, define $r = K - k_0 \leq (K+1)/2$ as well as two non-negative vectors $a^+, a^- \in R_+^{K+1}$,

$$a^+ = (0, ..., 0, a_{k_0+1}, ..., a_K),$$

and

$$a^- = (-a_0, ..., -a_{k_0}, 0, ..., 0).$$

The proof proceeds in two steps.

Step 1: Show that a^+ majorizes a^-. This is done by showing that (i) $\sum_{k=0}^{K} a_k = 0$ and (ii) that the sum of any number of elements sorted in descending order in a^+ is at least as large as the sum of the same number of elements also sorted in descending order in a^-, ie that for $m \leq r-1$, $-\sum_{k=0}^{m} a_k \leq \sum_{k=K-m}^{K} a_k$. For (i), use the definition of a_k. For (ii), use an induction argument. Take $m = 0$ and suppose to the contrary that $-a_0 > a_K$. Then convexity gives

$$a_K - a_{K-1} \geq a_1 - a_0$$

or, rearranging,

$$-a_1 - a_{K-1} \geq -a_0 - a_K > 0.$$

Thus, $-a_1 > a_{K-1}$. Repeating the argument gives $-a_k > a_{K-k}$, in which case it must be that $\sum_{k=0}^{K} a_k < \sum_{k=0}^{r-1} (a_k + a_{K-k}) < 0$, a contradiction. Now assume that $-\sum_{k=0}^{m} a_k \leq \sum_{k=K-m}^{K} a_k$ for $m < r-1$, but that $-\sum_{k=0}^{m+1} a_k > \sum_{k=K-m-1}^{K} a_k$. Then, it must be the case that $-a_{m+1} > a_{K-m-1}$. A similar argument to the one above implies a contradiction that the a_k cannot sum to zero.

Step 2: Use the fact that the function $f : R_+^{K+1} \rightarrow R_+$, $f(a) = \sum_{k=0}^{K} a_k^3$, is a Schur-convex function (see Marshall and Olkin, 1981). Schur-convex functions have the property that they preserve the ordering of majorizations, ie, if a^+ majorizes a^-, then $f(a^+) \geq f(a^-)$, or

$$\sum_{k=0}^{K} \left(a_k^+\right)^3 \geq \sum_{k=0}^{K} \left(a_k^-\right)^3.$$

It follows directly that $\sum_{k=0}^{K} a_k^3 \geq 0$. ■

References:

Albuquerque, R., 2012, Skewness in Stock Returns: Reconciling the Evidence on Firm Versus Aggregate Returns, *Review of Financial Studies*, doi: 10.1093/rfs/hhr144.

Marshall, A. W., and Olkin, I., 1981, Inequalities via Majorization–An Introduction, Technical Report No. 172, Stanford University.